Critical potentials of the eigenvalues and eigenvalue gaps of Schrödinger operators

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CRITICAL POTENTIALS OF THE EIGENVALUES AND EIGENVALUE GAPS OF SCHRÖDINGER OPERATORS

AHMAD EL SOUFI AND NAZIH MOUKADEM

Abstract. Let M be a compact Riemannian manifold with or without boundary, and let $-\Delta$ be its Laplace-Beltrami operator. For any bounded scalar potential q, we denote by $\lambda_i(q)$ the i-th eigenvalue of the Schrödinger type operator $-\Delta + q$ acting on functions with Dirichlet or Neumann boundary conditions in case $\partial M \neq \emptyset$. We investigate critical potentials of the eigenvalues λ_i and the eigenvalue gaps $G_{ij} = \lambda_j - \lambda_i$ considered as functionals on the set of bounded potentials having a given mean value on M. We give necessary and sufficient conditions for a potential q to be critical or to be a local minimizer or a local maximizer of these functionals. For instance, we prove that a potential $q \in L^\infty(M)$ is critical for the functional λ_2 if and only if, q is smooth, $\lambda_2(q) = \lambda_3(q)$ and there exist second eigenfunctions $f_1, \ldots, f_k$ of $-\Delta + q$ such that $\Sigma_j f_j^2 = 1$. In particular, λ_2 (as well as any λ_i) admits no critical potentials under Dirichlet Boundary conditions. Moreover, the functional λ_2 never admits locally minimizing potentials.

1. Introduction and Statement of main Results

Let M be a compact connected Riemannian manifold of dimension d, possibly with nonempty boundary ∂M, and let $-\Delta$ be its Laplace-Beltrami operator acting on functions with, in the case where $\partial M \neq \emptyset$, Dirichlet or Neumann boundary conditions. In all the sequel, as soon as the Neumann Laplacian will be considered, the boundary of M will be assumed to be sufficiently regular (e.g. C^1, but weaker regularity assmptions may suffice, see [3]) in order to guarantee the compactness of the embedding $H^1(M) \hookrightarrow L^2(M)$ and, hence, the compactness of the resolvent of the Neumann Laplacian (note that it is well known, using standard arguments like in [14, p.89], that compactness results for Sobolev spaces on Euclidean domains remain valid in the Riemannian setting).

For any bounded real valued potential q on M, the Schrödinger type operator $-\Delta + q$ has compact resolvent (see [16, Theorem IV.3.17] and observe that a bounded q leads to a relatively compact operator with respect to $-\Delta$). Therefore, its spectrum consists of a nondecreasing and unbounded sequence of eigenvalues with finite multiplicities:

$$Spec(-\Delta + q) = \{\lambda_1(q) < \lambda_2(q) \leq \lambda_3(q) \leq \cdots \leq \lambda_i(q) \leq \cdots\}.$$

Each eigenvalue $\lambda_i(q)$ can be considered as a (continuous) function of the potential $q \in L^\infty(M)$ and there are both physical and mathematical motivations to study existence and properties of extremal potentials of the

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functionals λ_i as well as of the differences, called gaps, between them. A very rich literature is devoted to the existence and the determination of maximizing or minimizing potentials for the eigenvalues (especially the fundamental one, λ_1) and the eigenvalue gaps (especially the first one, $\lambda_2 - \lambda_1$) under various constraints often motivated by physical considerations (see, for instance, [1, 2, 4, 6, 7, 10, 11, 12, 13, 17, 19] and the references therein). Note that, since the function λ_i commutes with constant translations, that is, $\lambda_i(q+c) = \lambda_i(q) + c$, such constraints are necessary.

Our aim in this paper is to investigate critical points, including "local minimizers" and "local maximizers", of the eigenvalue functionals $q \to \lambda_i(q)$ and the eigenvalue gap functionals $q \to \lambda_j(q) - \lambda_i(q)$, the potentials q being subjected to the constraint that their mean value (or, equivalently, their integral) over M is fixed. All along this paper, the mean value of an integrable function q will be denoted $\bar{q}$, that is,

$$\bar{q} = \frac{1}{V(M)} \int_M q\, dv,$$

$V(M)$ and dv being respectively the Riemannian volume and the Riemannian volume element of M.

Actually, most of the results below can be extended, modulo some slight changes, to the case where this constraint is replaced by the more general one

$$\int_M F(q) dv = \text{constant},$$

where $F : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $F'(x) \neq 0$ if $x \neq 0$, like $F(x) = |x|^\alpha$ or $F(x) = x|x|^{\alpha-1}$ with $\alpha \geq 1$. However, for simplicity and clarity reasons, we preferred to focus only on the mean value constraint. Therefore, we fix a constant $c \in \mathbb{R}$ and consider the functionals

$$\lambda_i : q \in L^\infty_c(M) \mapsto \lambda_i(q) \in \mathbb{R},$$

where $L^\infty_c(M) = \left\{ q \in L^\infty(M) \;\middle|\; \bar{q} = c \right\}$. The tangent space to $L^\infty_c(M)$ at any point q is given by

$$L^\infty_*(M) := \left\{ u \in L^\infty(M) \mid \int_M u\, dv = 0 \right\}.$$

1.1. **Critical potentials of the eigenvalue functionals.**

Since it is always nondegenerate, the first eigenvalue gives rise to a differentiable functional in the sense that, for any $q \in L^\infty_c(M)$ and any $u \in L^\infty_*(M)$, the function $t \mapsto \lambda_1(q + tu)$ is differentiable in t. A potential $q \in L^\infty_c(M)$ will be termed *critical* for this functional if $\frac{d}{dt}\lambda_1(q+tu)\big|_{t=0} = 0$ for any $u \in L^\infty_*(M)$.

In the case of empty boundary or of Neumann boundary conditions, the constant function 1 belongs to the domain of the operator $-\Delta + q$ and one obtains, as a consequence of the min-max principle, that the constant potential c is a global maximizer of λ_1 over $L^\infty_c(M)$ (see also [6] and [13]). Constant potential c is actually the only critical one for λ_1. On the other hand, under Dirichlet boundary conditions, the functional λ_1 admits no critical potentials in $L^\infty_c(M)$. Indeed, we have the following

Theorem 1.1. (1) *Assume that either $\partial M = \emptyset$ or $\partial M \neq \emptyset$ and Neumann boundary conditions are imposed. Then, for any potential q in $L^\infty_c(M)$, we have*

$$\lambda_1(q) \leq \lambda_1(c) = c,$$

where the equality holds if and only if $q = c$. Moreover, the constant potential c is the only critical one of the functional λ_1 over $L^\infty_c(M)$.

(2) *Assume that $\partial M \neq \emptyset$ and that Zero Dirichlet boundary conditions are imposed. Then the functional λ_1 does not admit any critical potential in $L^\infty_c(M)$.*

Higher eigenvalues are continuous but not differentiable in general. Nevertheless, perturbation theory enables us to prove that, for any function $u \in L^\infty(M)$, the function $t \mapsto \lambda_i(q + tu)$ admits left and right derivatives at $t = 0$ (see section 2.2). A generalized notion of criticality can be naturally defined as follows :

Definition 1.1. *A potential q is said to be critical for the functional λ_i if, for any $u \in L^\infty_*(M)$, the left and right derivatives of $t \mapsto \lambda_i(q + tu)$ at $t = 0$ have opposite signs, that is*

$$\frac{d}{dt}\lambda_i(q+tu)\Big|_{t=0^+} \times \frac{d}{dt}\lambda_i(q+tu)\Big|_{t=0^-} \leq 0.$$

It is immediate to check that q is critical for λ_i if and only if, for any $u \in L^\infty_*(M)$, one of the two following inequalities holds :

$$\lambda_i(q + tu) \leq \lambda_i(q) + o(t) \quad \text{as } t \to 0$$

or

$$\lambda_i(q + tu) \geq \lambda_i(q) + o(t) \quad \text{as } t \to 0.$$

In all the sequel, we will denote by $E_i(q)$ the eigenspace corresponding to the i-th eigenvalue $\lambda_i(q)$ whose dimension coincides with the number of indices $j \in \mathbb{N}$ such that $\lambda_j(q) = \lambda_i(q)$.

As for the first eigenvalue, the functionals λ_i, $i \geq 2$, admit no critical potentials under Dirichlet boundary conditions.

Theorem 1.2. *Assume that $\partial M \neq \emptyset$ and that Zero Dirichlet boundary conditions are imposed. Then, $\forall i \in \mathbb{N}^*$, the functional λ_i does not admit any critical potential in $L^\infty_c(M)$.*

Under the two remaining boundary conditions, the following theorem gives a necessary condition for a potential q to be critical for the functional λ_i. This condition is also sufficient for the indices i such that $\lambda_i(q) > \lambda_{i-1}(q)$ or $\lambda_i(q) < \lambda_{i+1}(q)$, which means that $\lambda_i(q)$ is the first one or the last one in a cluster of equal eigenvalues.

Theorem 1.3. *Assume that either $\partial M = \emptyset$ or $\partial M \neq \emptyset$ and Neumann boundary conditions are imposed. Let i be a positive integer.*

If $q \in L^\infty_c(M)$ is a critical potential of the functional λ_i, then q is smooth and there exists a finite family of eigenfunctions $f_1, \dots, f_k$ in $E_i(q)$ such that $\sum_{1\leq j\leq k} f_j^2 = 1$.

Reciprocally, if $\lambda_i(q) > \lambda_{i-1}(q)$ or $\lambda_i(q) < \lambda_{i+1}(q)$, and if there exists a family of eigenfunctions $f_1, \ldots, f_k \in E_i(q)$ such that $\sum_{1\le j\le k} f_j^2 = 1$, then q is a critical potential of the functional λ_i.

Note that the identity $\sum_{1\le j\le k} f_j^2 = 1$, with $f_1, \ldots, f_k \in E_i(q)$, immediately implies another one (that we obtain from $\Delta \sum_{1\le j\le k} f_j^2 = 0$):

$$q = \lambda_i(q) - \sum_{1\le j\le k} |\nabla f_j|^2,$$

from which we can deduce the smoothness of q.

Remark 1.1. *1. The identity $\sum_{1\le j\le k} f_j^2 = 1$ with $-\Delta f_j + q f_j = \lambda_i(q) f_j$, means that the map $f = (f_1, \ldots, f_k)$ from M to the Euclidean sphere $\mathbb{S}^{k-1}$ is harmonic with energy density $|\nabla f|^2 = \lambda_i(q) - q$ (see [5]). Hence, a necessary (and sometime sufficient) condition for a potential q to be critical for the functional λ_i is that the function $\lambda_i(q) - q$ is the energy density of a harmonic map from M to a Euclidean sphere.*

2. If one replaces the constraint on the mean value $\frac{1}{V(M)} \int_M q dv = c$ by the general constraint $\int_M F(q) dv = c$, then the necessary and sufficient condition $\sum_{1\le j\le k} f_j^2 = 1$ of Theorem 1.3 becomes (even under Dirichlet boundary conditions) $\sum_{1\le j\le k} f_j^2 = F'(q)$. In particular, q is a critical potential of the functional λ_1 if and only if $F'(q) \ge 0$ and $F'(q)^{\frac{1}{2}}$ is a first eigenfunction of $-\Delta + q$, see [1, 12] for a discussion of the case $F(q) = |q|^\alpha$.

Under each one of the boundary conditions we consider, a constant function can never be an eigenfunction associated to an eigenvalue $\lambda_i(q)$ with $i \ge 2$. Hence, an immediate consequence of Theorem 1.3 is the following

Corollary 1.1. *If $q \in L_c^\infty(M)$ is a critical potential of the functional λ_i with $i \ge 2$, then the eigenvalue $\lambda_i(q)$ is degenerate, that is $\lambda_i(q) = \lambda_{i-1}(q)$ or $\lambda_i(q) = \lambda_{i+1}(q)$*

If $\{f_1, \ldots, f_k\}$ is an L^2-orthonormal basis of $E_i(-\Delta)$, then the function $\sum_{1\le j\le k} f_j^2$ is invariant under the isometry group of M. Indeed, for any isometry ρ of M, $\{f_1 \circ \rho, \ldots, f_k \circ \rho\}$ is also an orthonomal basis of $E_i(-\Delta)$ and then, there exists a matrix $A \in O(d)$ such that $(f_1 \circ \rho, \ldots, f_d \circ \rho) = A.(f_1, \ldots, f_d)$. In particular, if M is homogeneous, that is, the isometry group acts transitively on M, then $\sum_{1\le j\le k} f_j^2$ would be constant. Another consequence of Theorem 1.3 is then the following

Corollary 1.2. *If M is homogeneous, then constant potentials are critical for all the functionals λ_i such that $\lambda_i(-\Delta) < \lambda_{i+1}(-\Delta)$ or $\lambda_i(-\Delta) > \lambda_{i-1}(-\Delta)$.*

Recall that Euclidean spheres, projective spaces and flat tori are examples of homogeneous Riemannian spaces.

A potential $q \in L_c^\infty(M)$ is said to be a *local minimizer* (resp. *local maximizer*) of the functional λ_i (in a weak sense) if, for any $u \in L_*^\infty(M)$, the function $t \mapsto \lambda_i(q + tu)$ admits a local minimum (resp. maximum) at $t = 0$. The result of Corollary 1.1 takes the following more precise form in the case of a local minimizer or maximizer.

Theorem 1.4. *Let $q \in L^\infty_c(M)$ and $i \geq 2$.*

(1) *If q is a local minimizer of the functional λ_i, then $\lambda_i(q) = \lambda_{i-1}(q)$.*
(2) *If q is a local maximizer of the functional λ_i, then $\lambda_i(q) = \lambda_{i+1}(q)$.*

Since the first eigenvalue is simple, we always have $\lambda_2(q) > \lambda_1(q)$. The previous results, applied to the functional λ_2 can be summarized as follows.

Corollary 1.3. *Assume that either $\partial M = \emptyset$ or $\partial M \neq \emptyset$ and Neumann boundary conditions are imposed. A potential $q \in L^\infty_c(M)$ is critical for the functional λ_2 if and only if, q is smooth, $\lambda_2(q) = \lambda_3(q)$ and there exist eigenfunctions $f_1, \dots, f_k$ in $E_2(q)$ such that $\sum_{1\leq j\leq k} f_j^2 = 1$.*

Moreover, the functional λ_2 admits no local minimizers in $L^\infty_c(M)$.

In [6], Ilias and the first author have proved that, under some hypotheses on M, satisfied in particular by compact rank-one symmetric spaces, irreducible homogeneous Riemannian spaces and some flat tori, the constant potential c is a global maximizer of λ_2 over $L^\infty_c(M)$. In [8, 9], they studied the critical points of λ_i considered as a functional on the set of Riemannian metrics of fixed volume on M.

1.2. **Critical potentials of the eigenvalue gaps functionals.**

We consider now the eigenvalue gaps functionals $q \mapsto G_{ij}(q) = \lambda_j(q) - \lambda_i(q)$, where i and j are two distinct positive integers, and define their critical potentials as in Definition 1.1. These functionals are invariant under translations, that is $G_{ij}(q+c) = G_{ij}(q)$. Therefore, critical potentials of G_{ij} with respect to fixed mean value deformations are also critical with respect to arbitrary deformations.

Theorem 1.5. *If $q \in L^\infty_c(M)$ is a critical potential of the gap functional $G_{ij} = \lambda_j - \lambda_i$, then there exist a finite family of eigenfunctions $f_1, \dots, f_k$ in $E_i(q)$ and a finite family of eigenfunctions $g_1, \dots, g_l$ in $E_j(q)$, such that $\sum_{1\leq p\leq k} f_p^2 = \sum_{1\leq p\leq l} g_p^2$.*

Reciprocally, if $\lambda_i(q) < \lambda_{i+1}(q)$ and $\lambda_j(q) > \lambda_{j-1}(q)$, and if there exist $f_1, \dots, f_k$ in $E_i(q)$ and $g_1, \dots, g_l$ in $E_j(q)$ such that $\sum_{1\leq p\leq k} f_p^2 = \sum_{1\leq p\leq l} g_p^2$, then q is a critical potential of G_{ij}.

In the particular case of the gap between two consecutive eigenvalues, we have the following

Corollary 1.4. *A potential $q \in L^\infty_c(M)$ is critical for the gap functional $G_{i,i+1} = \lambda_{i+1} - \lambda_i$ if and only if, either $\lambda_{i+1}(q) = \lambda_i(q)$, or there exist a family of eigenfunctions $f_1, \dots, f_k$ in $E_i(q)$ and a family of eigenfunctions $g_1, \dots, g_l$ in $E_{i+1}(q)$, such that $\sum_{1\leq p\leq k} f_p^2 = \sum_{1\leq p\leq l} g_p^2$.*

Remark 1.2. *The characterization of critical potentials of G_{ij} given in Theorem 1.5 remains valid under the constraint $\int_M F(q)dv = c$.*

An immediate consequence of Theorem 1.5 is the following

Corollary 1.5. *Let $q \in L^\infty_c(M)$ be a critical potential of the gap functional $G_{ij} = \lambda_j - \lambda_i$. If $\lambda_i(q)$ (resp. $\lambda_j(q)$) is nondegenerate, then $\lambda_j(q)$ (resp. $\lambda_i(q)$) is degenerate.*

The following is an immediate consequence of the discussion above concerning homogeneous Riemannian manifolds.

Corollary 1.6. *If M is a homogeneous Riemannian manifold, then, for any positive integer i, constant potentials are critical points of the gap functional $G_{i,i+1} = \lambda_{i+1} - \lambda_i$.*

Potentials q such that $\lambda_{i+1}(q) = \lambda_i(q)$ are of course global minimizers of the gap functional $G_{i,i+1}$. These potentials are also the only local minimizers of $G_{i,i+1}$. Indeed, we have the following

Theorem 1.6. *If $q \in L^\infty_c(M)$ is a local minimizer of the gap functional $G_{ij} = \lambda_j - \lambda_i$, then, either $\lambda_i(q) = \lambda_{i+1}(q)$, or $\lambda_j(q) = \lambda_{j-1}(q)$. If q is a local maximizer of G_{ij}, then, either $\lambda_i(q) = \lambda_{i-1}(q)$, or $\lambda_j(q) = \lambda_{j+1}(q)$.*

In particular, q is a local minimizer of the gap functional $G_{i,i+1} = \lambda_{i+1} - \lambda_i$ if and only if $G_{i,i+1}(q) = 0$.

Finally, let us apply the results of this section to the first gap $G_{1,2}$.

Corollary 1.7. *A potential $q \in L^\infty_c(M)$ is critical for the gap functional $G_{1,2} = \lambda_2 - \lambda_1$ if and only if $\lambda_2(q)$ is degenerate and there exists a family of eigenfunctions $g_1, \dots, g_l$ in $E_2(q)$ such that $\sum_{1\le j\le l} g_j^2 = f^2$, where f is a basis of $E_1(q)$.*

The functional $G_{1,2}$ does not admit any local minimizer in $L^\infty_c(M)$.

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2. Proof of Results

2.1. **Variation Formula and proof of Theorem 1.1.** Given on M a potential q and a function $u \in L^\infty(M)$, we consider the family of operators $-\Delta + q + tu$. Suppose that $\Lambda(t)$ is a differentiable family of eigenvalues of $-\Delta + q + tu$ and that f_t is a differentiable family of corresponding normalized eigenfunctions, that is, $\forall t$,

$$(-\Delta + q + tu) f_t = \Lambda(t) f_t,$$

and

$$\int_M f_t^2 dv = 1,$$

with $f_t\big|_{\partial M} = 0$ or $\frac{\partial f_t}{\partial \nu}\big|_{\partial M} = 0$ if $\partial M \neq \emptyset$. The following formula, giving the derivative of Λ, is already known at least in the case of Euclidean domains with Dirichlet boundary conditions.

Proposition 2.1.

$$\Lambda'(0) = \int_M u f_0^2 dv.$$

Proof. First, we have, for all t,

$$\Lambda(t) = \Lambda(t) \int_M (f_t)^2 dv = \int_M f_t(-\Delta + q + tu) f_t \, dv.$$

Differentiating at $t = 0$, we get

$$\Lambda'(0) = \frac{d}{dt}\left(\int_M f_t(-\Delta + q) f_t \, dv + t \int_M u (f_t)^2 dv \right)\Big|_{t=0}.$$

Now, noticing that the function $\frac{d}{dt}f_t\big|_{t=0}$ satisfies the same boundary conditions as f_0 in case $\partial M \neq \emptyset$, and using integration by parts, we obtain

$$\begin{aligned}\frac{d}{dt}\int_M f_t(-\Delta+q)f_t\,dv\Big|_{t=0} &= 2\int_M(-\Delta+q)f_0\,\frac{d}{dt}f_t\Big|_{t=0}dv\\ &= 2\Lambda(0)\int_M f_0\,\frac{d}{dt}f_t\Big|_{t=0}dv\\ &= \Lambda(0)\,\frac{d}{dt}\int_M f_t^2dv\Big|_{t=0}=0.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\frac{d}{dt}\Big(t\int_M uf_t^2dv\Big)\Big|_{t=0} &= \int_M uf_0^2dv+\Big(t\int_M u\frac{d}{dt}f_t^2dv\Big)\Big|_{t=0}\\ &= \int_M uf_0^2v_g.\end{aligned}$$

Finally, $\Lambda'(0)=\int_M uf_0^2dv$. □

Proof. (of Theorem 1.1.) (i) First, let us show that constant potentials are maximizing for λ_1. Indeed, let c be a constant potential and let q be an arbitrary one in $L^\infty_c(M)$. From the variational characterization of $\lambda_1(-\Delta+q)$ in the case $\partial M=\emptyset$ as well as in the case of Neumann boundary conditions, we get

$$\begin{aligned}\lambda_1(-\Delta+q) &= \inf_{f\in H^1(M)}\frac{\int_M(|\nabla f|^2+qf^2)dv}{\|f\|^2_{L^2(M)}}\\ &\leq \frac{\int_M(|\nabla 1|^2+q1^2)dv}{\|1\|^2_{L^2(M)}}=\frac{\int_M q\,dv}{V(M)}=c.\end{aligned}$$

Hence, $\lambda_1(q)\leq\lambda_1(c)$ and the constant potential c maximizes the functional λ_1 on $L^\infty_c(M)$. In particular, constant potentials are critical for this functional.

Now, suppose that $q\in L^\infty_c(M)$ is a critical potential for λ_1. For any $u\in L^\infty_*(M)$, we consider a differentiable family f_t of normalized eigenfunctions corresponding to the first eigenvalue of $(-\Delta+q+tu)$ and apply the variation formula above to obtain:

$$\frac{d}{dt}\lambda_1(q+tu)\Big|_{t=0}=\int_M uf_0^2\,dv.$$

Hence, $\int_M uf_0^2dv=0$ for any $u\in L^\infty_*(M)$, which implies that f_0 is constant on M. Since $(-\Delta+q)f_0=qf_0=\lambda_1(q)f_0$, the potential q must be constant on M.

(ii) Let f_0 be the first nonnegative Dirichlet eigenfunction of $-\Delta+q$ satisfying $\int_M f_0^2\,dv=1$. The function $u=V(M)f_0^2-1$ belongs to $L^\infty_*(M)$ and we have

$$\frac{d}{dt}\lambda_1(q+tu)\Big|_{t=0}=\int_M uf_0^2\,dv=V(M)\int_M f_0^4\,dv-1>0,$$

where the last inequality comes from Cauchy-Schwarz inequality and the fact that f_0 is not constant (recall that $f_0\big|_{\partial M}=0$). Therefore, the potential q is not critical for λ_1. □

2.2. **Characterization of critical potentials.** Let i be a positive integer and let $m \geq 1$ be the dimension of the eigenspace $E_i(q)$ associated to the eigenvalue $\lambda_i(q)$. For any function $u \in L^\infty_*(M)$, perturbation theory of unbounded self-adjoint operators (see for instance Kato's book [16]) that we apply to the one parameter family of operators $-\Delta + q + tu$, tells us that, there exists a family of m eigenfunctions $f_{1,t}, \ldots, f_{m,t}$ associated with a family of m (non ordered) eigenvalues $\Lambda_1(t), \ldots, \Lambda_m(t)$ of $-\Delta + q + tu$, all depending analytically in t in some interval $(-\varepsilon, \varepsilon)$, and satisfying

- $\Lambda_1(0) = \cdots = \Lambda_m(0) = \lambda_i(q)$,
- $\forall t \in (-\varepsilon, \varepsilon)$, the m functions $f_{1,t}, \ldots, f_{m,t}$ are orthonormal in $L^2(M)$.

From this, one can easily deduce the existence of two integers $k \leq m$ and $l \leq m$, and a small $\delta > 0$ such that

$$\lambda_i(q + tu) = \begin{cases} \Lambda_k(t) \text{ if } t \in (-\delta, 0) \\ \\ \Lambda_l(t) \text{ if } \ t \in (0, \delta). \end{cases}$$

Hence, the function $t \mapsto \lambda_i(q + tu)$ admits a left sided and a right sided derivatives at $t = 0$ with

$$\frac{d}{dt}\lambda_i(q + tu)\Big|_{t=0^-} = \Lambda'_k(0) = \int_M u f_{k,0}^2 dv$$

and

$$\frac{d}{dt}\lambda_i(q + tu)\Big|_{t=0^+} = \Lambda'_l(0) = \int_M u f_{l,0}^2 dv.$$

To any function $u \in L^\infty_*(M)$ and any integer $i \in \mathbb{N}$, we associate the quadratic form Q^i_u on $E_i(q)$ defined by

$$Q^i_u(f) = \int_M u f^2 dv.$$

The corresponding symmetric linear transformation $L^i_u : E_i(q) \to E_i(q)$ is given by

$$L^i_u(f) = P_i(uf),$$

where $P_i : L^2(M) \to E_i(q)$ is the orthogonal projection of $L^2(M)$ onto $E_i(q)$.

It follows immediately that

Proposition 2.2. *If the potential q is critical for the functional λ_i, then, $\forall u \in L^\infty_*(M)$, the quadratic form $Q^i_u(f) = \int_M u f^2 dv$ is indefinite on the eigenspace $E_i(q)$.*

The following lemma enables us to establish a converse to this proposition.

Lemma 2.1. *$\forall k, l \leq m$, we have*

$$\int_M u f_{k,0} f_{l,0}\, dv = \begin{cases} 0 \textit{ if } \ k \neq l \\ \\ \Lambda'_k(0) \textit{ if } \ k = l. \end{cases}$$

In other words, $\Lambda'_1(0), \ldots, \Lambda'_m(0)$ are the eigenvalues of the symmetric linear transformation $L^i_u : E_i(q) \to E_i(q)$ and the functions $f_{1,0}, \ldots, f_{m,0}$ constitute an orthonormal eigenbasis of L^i_u.

Proof. Differentiating at $t = 0$ the equality $(-\Delta + q + tu)f_{k,t} = \Lambda_k(t)f_{k,t}$, we obtain

$$uf_{k,0} + (-\Delta + q)\frac{d}{dt}f_{k,t}\Big|_{t=0} = \Lambda'_k(0)f_{k,0} + \Lambda_k(0)\frac{d}{dt}f_{k,t}\Big|_{t=0},$$

and then,

$$\begin{aligned}\int_M uf_{k,0}f_{l,0}\,dv = \Lambda'_k(0)\int_M f_{k,0}f_{l,0}\,dv \quad &+ \quad \Lambda_k(0)\int_M f_{l,0}\frac{d}{dt}f_{k,t}\Big|_{t=0}dv \\ &- \quad \int_M f_{l,0}(-\Delta + q)\frac{d}{dt}f_{k,t}\Big|_{t=0}dv.\end{aligned}$$

Integration by parts gives, after noticing that $\Lambda_k(0) = \Lambda_l(0) = \lambda_i(q)$ and that the functions $\frac{d}{dt}f_{k,t}\big|_{t=0}$ satisfy the considered boundary conditions,

$$\begin{aligned}\int_M f_{l,0}(-\Delta + q)\frac{d}{dt}f_{k,t}\Big|_{t=0}dv &= \int_M \frac{d}{dt}f_{k,t}\Big|_{t=0}(-\Delta + q)f_{l,0}\,dv \\ &= \Lambda_k(0)\int_M f_{l,0}\frac{d}{dt}f_{k,t}\Big|_{t=0}dv,\end{aligned}$$

and finally,

$$\int_M uf_{k,0}f_{l,0}\,dv = \Lambda'_k(0)\int_M f_{k,0}f_{l,0}\,dv = \Lambda'_k(0)\delta_{kl}.$$

□

Proposition 2.3. *Assume that $\lambda_i(q) > \lambda_{i-1}(q)$ or $\lambda_i(q) < \lambda_{i+1}(q)$. Then the following conditions are equivalent:*

i) *the potential q is critical for λ_i*

ii) *$\forall u \in L^\infty_*(M)$, the quadratic form $Q^i_u(f) = \int_M uf^2dv$ is indefinite on the eigenspace $E_i(q)$.*

iii) *$\forall u \in L^\infty_*(M)$, the linear transformation L^i_u admits eigenvalues of both signs.*

Proof. Conditions (ii) and (iii) are clearly equivalent and the fact that (i) implies (ii) was established in Proposition 2.2. Let us show that (iii) implies (i). Assume that $\lambda_i(q) > \lambda_{i-1}(q)$ and let $u \in L^\infty_*(M)$ and $\Lambda_1(t), \dots, \Lambda_m(t)$ be as above. For small t, we will have, for continuity reasons, $\forall k \le m$, $\Lambda_k(t) > \lambda_{i-1}(q + tu)$ and then, $\lambda_i(q + tu) \le \Lambda_k(t)$. Since $\lambda_i(q + tu) \in \{\Lambda_1(t), \dots, \Lambda_m(t)\}$, we get

$$\lambda_i(q + tu) = \min_{k \le m}\Lambda_k(t).$$

It follows that

$$\frac{d}{dt}\lambda_i(q + tu)\Big|_{t=0^-} = \max_{k \le m}\Lambda'_k(0)$$

and

$$\frac{d}{dt}\lambda_i(q + tu)\Big|_{t=0^+} = \min_{k \le m}\Lambda'_k(0).$$

Thanks to Lemma 2.1, Condition (iii) implies that $\min_{k \le m}\Lambda'_k(0) \le 0 \le \max_{k \le m}\Lambda'_k(0)$ which implies the criticality of q.

The case $\lambda_i(q) < \lambda_{i+1}(q)$ can be treated in a similar manner. □

2.3. **Proof of Theorems 1.2 and 1.3.** Let q be a potential in $L^\infty_c(M)$. To prove Theorem 1.2 we first notice that, since $f\big|_{\partial M} = 0$ for any $f \in E_i(q)$, the constant function 1 does not belong to the vector space F generated in $L^2(M)$ by $\{f^2\,; f \in E_i(q)\}$. Hence, there exists a function u orthogonal to F and such that $\langle u, 1\rangle_{L^2(M)} < 0$. The function $u_0 = u - \bar{u}$ belongs to $L^\infty_*(M)$ and the quadratic form $Q^i_{u_0}(f) = \int_M u_0 f^2 dv = -\bar{u}\|f\|^2_{L^2(M)}$ is positive definite on $E_i(q)$. Hence, the potential q is not critical for λ_i (see Proposition 2.2).

The proof of Theorem 1.3 follows directly from the two propositions above and the following lemma.

Lemma 2.2. *Let i be a positive integer. The two following conditions are equivalent:*

i) $\forall u \in L^\infty_*(M)$*, the quadratic form* $Q^i_u(f) = \int_M u f^2 dv$ *is indefinite on the eigenspace* $E_i(q)$.

ii) *there exists a family of eigenfunctions* $f_1, \ldots, f_k$ *in* $E_i(q)$ *such that* $\sum_{1\le j\le k} f_j^2 = 1$.

Proof. To see that (i) implies (ii) we introduce the convex cone C generated in $L^2(M)$ by the set $\{f^2\,; f \in E_i(q)\}$, that is $C = \{\sum_{j\in J} f_j^2\,; f_j \in E_i(q), J \subset \mathbb{N},\ J \text{ is finite}\}$. Condition (ii) is then equivalent to the fact that the constant function 1 belongs to C. Let us suppose, for a contradiction, that $1 \notin C$. Then, applying classical separation theorems (in the finite dimensional vector subspace of $L^2(M)$ generated by $\{f^2\,; f \in E_i(q)\}$ and 1, see [18]), we prove the existence of a function $u \in L^2(M)$ such that $\bar{u} = \frac{1}{V(M)}\int_M u\cdot 1\,dv < 0$ and $\int_M u f^2 dv \ge 0$ for any $f \in C$. Hence, the function $u_0 = u - \bar{u}$ belongs to $L^\infty_*(M)$ and satisfies, $\forall f \in E_i(q)$,

$$Q^i_{u_0}(f) = \int_M u f^2 dv - \frac{1}{V(M)}\int_M u\,dv \int_M f^2 dv \ge -\bar{u}\|f\|^2_{L^2(M)}.$$

The quadratic form $Q^i_{u_0}$ is then positive definite which contradicts (i) (see Proposition 2.2).

Reciprocally, the existence of $f_1, \ldots, f_k$ in $E_i(q)$ satisfying $\sum_{1\le j\le k} f_j^2 = 1$ implies that, $\forall u \in L^\infty_*(M)$,

$$\sum_{j\le k} Q^i_u(f_j) = \sum_{j\le k}\int_M u f_j^2 dv = \int_M u = 0,$$

which implies that the quadratic form Q^i_u is indefinite on $E_i(q)$. $\square$

Finally, let us check that the condition $\sum_{1\le j\le k} f_j^2 = 1$, with $f_j \in E_i(q)$, implies that q is smooth. Indeed, since $q \in L^\infty(M)$, we have, for any eigenfunction $f \in E_i(q)$, $\Delta f \in L^2(M)$ and then, $f \in H^{2,2}(M)$. Using standard regularity theory and Sobolev embeddings (see, for instance, [15]), we obtain by an elementary iteration, that $f \in H^{2,p}(M)$ for some $p > n$, and, then, $f \in C^1(M)$. From $\sum_{1\le j\le k} f_j^2 = 1$ and $\Delta \sum_{1\le j\le k} f_j^2 = 0$, we get

$$q = \lambda_i(q) - \sum_{1\le j\le k} |\nabla f_j|^2,$$

which implies that q is continuous. Again, elliptic regularity theory tells us that the eigenfunctions of $-\Delta + q$ are actually smooth, and, hence, q is smooth.

2.4. **Proof of Theorem 1.4.** Assume that the potential q is a local minimizer of the functional λ_i on $L^\infty_c(M)$ and let us suppose for a contradiction that $\lambda_i(q) > \lambda_{i-1}(q)$. Let u be a function in $L^\infty_*(M)$ and let $\Lambda_1(t), \dots, \Lambda_m(t)$ be a family of m eigenvalues of $-\Delta + q + tu$, where m is the multiplicity of $\lambda_i(q)$, depending analytically in t and such that $\Lambda_1(0) = \cdots = \Lambda_m(0) = \lambda_i(q)$. For continuity reasons, we have, for sufficiently small t and any $k \le m$, $\Lambda_k(t) > \lambda_{i-1}(q + tu)$. Hence, $\forall k \le m$ and $\forall t$ sufficiently small,

$$\Lambda_k(t) \ge \lambda_i(q + tu) \ge \lambda_i(q) = \Lambda_k(0).$$

Consequently, $\forall k \le m$, $\Lambda'_k(0) = 0$. Applying Lemma 2.1 above we deduce that the symmetric linear transformation L^i_u and then the quadratic form Q^i_u is identically zero on the eigenspace $E_i(q)$. Therefore, $\forall u \in L^\infty_*(M)$ and $\forall f \in E_i(q)$, we have $\int_M u f^2 v_g = 0$. In conclusion, $\forall f \in E_i(q)$, f is constant on M which is impossible for $i \ge 2$. The same arguments work to prove Assertion (ii).

2.5. **Proof of Theorem 1.5.** Let q be a potential and let i and j be two distinct positive integers such that $\lambda_i(q) \ne \lambda_j(q)$. We denote by m (resp. n) the dimension of the eigenspace $E_i(q)$ (resp. $E_j(q)$). Given a function u in $L^\infty_*(M)$, we consider, as above, m (resp. n) $L^2(M)$-orthonormal families of eigenfunctions $f_{1,t}, \dots, f_{m,t}$ (resp. $g_{1,t}, \dots, g_{n,t}$) associated with m (resp. n) families of eigenvalues $\Lambda_1(t), \dots, \Lambda_m(t)$ (resp. $\Gamma_1(t), \dots, \Gamma_n(t)$) of $-\Delta + q + tu$, all depending analytically in $t \in (-\varepsilon, \varepsilon)$, such that $\Lambda_1(0) = \cdots = \Lambda_m(0) = \lambda_i(q)$ (resp. $\Gamma_1(0) = \cdots = \Gamma_n(0) = \lambda_j(q)$). Hence, there exist four integers $k \le m$, $k' \le m$, $l \le n$ and $l' \le n$, such that

$$\begin{aligned} \frac{d}{dt}(\lambda_j - \lambda_i)(q + tu)\Big|_{t=0^-} &= \Gamma'_l(0) - \Lambda'_k(0) \\ &= \int_M u(g^2_{l,0} - f^2_{k,0})dv \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}(\lambda_j - \lambda_i)(q + tu)\Big|_{t=0^+} &= \Gamma'_{l'}(0) - \Lambda'_{k'}(0) \\ &= \int_M u(g^2_{l',0} - f^2_{k',0})dv. \end{aligned}$$

Recall that (Lemma 2.1) the eigenfunctions $f_{1,0}, \dots, f_{m,0}$ (resp. $g_{1,0}, \dots, g_{n,0}$) constitutes an $L^2(M)$-orthonormal basis of $E_i(q)$ (resp. $E_j(q)$) which diagonalizes the quadratic form Q^i_u (resp. Q^j_u). Therefore, the family $(f_{k,0} \otimes g_{l,0})_{k \le m, l \le n}$ constitutes a basis of the space $E_i(q) \otimes E_j(q)$ which diagonalizes the quadratic form $S^{i,j}_u$ given by

$$\begin{aligned} S^{i,j}_u(f \otimes g) &= \|f\|^2_{L^2(M)} Q^j_u(g) - \|g\|^2_{L^2(M)} Q^i_u(f) \\ &= \int_M u(\|f\|^2_{L^2(M)} g^2 - \|g\|^2_{L^2(M)} f^2)dv. \end{aligned}$$

The corresponding eigenvalues are $(\Gamma_l'(0)-\Lambda_k'(0))_{k\le m\,,l\le n}$. The criticality of q for $\lambda_j-\lambda_i$ then implies that this quadratic form admits eigenvalues of both signs, which means that it is indefinite.

On the other hand, in the case where $\lambda_i(q)<\lambda_{i+1}(q)$ and $\lambda_j(q)>\lambda_{j-1}(q)$, we have, as in the proof of Proposition 2.3, for sufficiently small t, $\lambda_i(q+tu)=\max_{k\le m}\Lambda_k(t)$ and $\lambda_j(q+tu)=\min_{l\le n}\Gamma_l(t)$, which yields

$$\begin{aligned}\frac{d}{dt}(\lambda_j-\lambda_i)(q+tu)\Big|_{t=0^-} &= \max_{l\le n}\Gamma_l'(0)-\min_{k\le m}\Lambda_k'(0)\\ &= \max_{k\le m\,,l\le n}(\Gamma_l'(0)-\Lambda_k'(0))\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}(\lambda_j-\lambda_i)(q+tu)\Big|_{t=0^+} &= \min_{l\le n}\Gamma_l'(0)-\max_{k\le m}\Lambda_k'(0)\\ &= \min_{k\le m\,,l\le n}(\Gamma_l'(0)-\Lambda_k'(0)).\end{aligned}$$

One deduces the following

Proposition 2.4. *If the potential $q\in L_c^\infty(M)$ is critical for the functional $G_{ij}=\lambda_j-\lambda_i$, then, $\forall u\in L_*^\infty(M)$, the quadratic form $S_u^{i,j}$ is indefinite on $E_i(q)\otimes E_j(q)$.*

Reciprocally, if $\lambda_i(q)<\lambda_{i+1}(q)$ and $\lambda_j(q)>\lambda_{j-1}(q)$, and if, $\forall u\in L_*^\infty(M)$, the quadratic form $S_u^{i,j}(g)$ is indefinite on $E_i(q)\otimes E_j(q)$, then q is a critical potential of the functional G_{ij}.

The following lemma will completes the proof of Theorem 1.5

Lemma 2.3. *The two following conditions are equivalent:*

i) *$\forall u\in L_*^\infty(M)$, the quadratic form $S_u^{i,j}$ is indefinite on $E_i(q)\otimes E_j(q)$.*
ii) *there exist a finite family of eigenfunctions $f_1,\dots,f_k$ in $E_i(q)$ and a finite family of eigenfunctions $g_1,\dots,g_l$ in $E_j(q)$, such that $\sum_{1\le p\le k}f_p^2=\sum_{1\le p\le l}g_p^2$.*

The proof of this lemma is similar to that of Lemma 2.2. Here, we consider the two convex cones C_i and C_j in $L^2(M)$ generated respectively by $\left\{f^2\,;\,f\in E_i(q)\,,f\ne 0\right\}$ and $\left\{g^2\,;\,g\in E_j(q)\,,g\ne 0\right\}$. Condition (ii) is then equivalent to the fact that these two cones admit a nontrivial intersection. As in the proof of Lemma 2.2, separation theorems enable us to prove that, if $C_i\cap C_j=\emptyset$, then there exists a function u such that $\int_M uf^2dv<0$ for any $f\in E_i(q)$, and $\int_M ug^2dv\ge 0$ for any $f\in E_j(q)$, which implies that $S_u^{i,j}$ is positive definite on $E_i(q)\otimes E_j(q)$. Since $S_1^{i,j}=0$, we have, $S_u^{i,j}=S_{u_0}^{i,j}$ with $u_0=u-\bar u\in L_*^\infty(M)$. Proposition 2.4 enables us to conclude.

Reciprocally, assume the existence of $f_1,\dots,f_k\in E_i(q)$ and $g_1,\dots,g_l\in E_j(q)$ satisfying $\sum_{1\le p\le k}f_p^2=\sum_{1\le p\le l}g_p^2$. Then, $\forall u\in L_*^\infty(M)$,

$$\sum_{1\le p\le k}\ \sum_{1\le p'\le l}S_u^{i,j}(f_p\otimes g_{p'})=\cdots=0,$$

which implies that $S_u^{i,j}$ is indefinite on $E_i(q)\otimes E_j(q)$.

2.6. **Proof of Theorem 1.6.** Let q be a local minimizer of $G_{ij} = \lambda_j - \lambda_i$ and let us suppose for a contradiction that $\lambda_i(q) < \lambda_{i+1}(q)$ and $\lambda_j(q) > \lambda_{j-1}(q)$. Given a function u in $L^\infty_*(M)$, we consider, as above, m (resp. n) families of eigenvalues $\Lambda_1(t), \dots, \Lambda_m(t)$ (resp. $\Gamma_1(t), \dots, \Gamma_n(t)$) of $-\Delta + q + tu$, with $m = \dim E_i(q)$ and $n = \dim E_j(q)$, such that $\Lambda_1(0) = \cdots = \Lambda_m(0) = \lambda_i(q)$ and $\Gamma_1(0) = \cdots = \Gamma_n(0) = \lambda_j(q)$. As in the proof of Theorem 1.4, we will have for sufficiently small t, $\lambda_i(q + tu) = \max_{k \le m} \Lambda_k(t)$ and $\lambda_j(q + tu) = \min_{l \le n} \Gamma_l(t)$. Hence, $\forall k \le m$ and $l \le n$,

$$\begin{aligned} \Gamma_l(t) - \Lambda_k(t) &\geq \lambda_j(q + tu) - \lambda_i(q + tu) = G_{ij}(q + tu) \\ &\geq G_{ij}(q) = \Gamma_l(0) - \Lambda_k(0). \end{aligned}$$

It follows that, $\forall k \le m$ and $l \le n$, $\Gamma_l'(0) - \Lambda_k'(0) = 0$ and, then, the quadratic form $S_u^{i,j}$ is identically zero on $E_i(q) \otimes E_j(q)$ (recall that $\Gamma_l'(0) - \Lambda_k'(0)$ are the eigenvalues of $S_u^{i,j}$). This implies that, $\forall f \in E_i(q)$ and $\forall g \in E_j(q)$, the function $\|f\|^2_{L^2(M)} g^2 - \|g\|^2_{L^2(M)} f^2$ is constant equal to zero (since its integral vanishes) which is clearly impossible unless $i = j$.

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